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Staggered ice-rule vertex model on the Kagomé lattice

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Abstract. The most general Pfaffian solution of the staggered ice-rule vertex model on the Kagomé lattice is given in this paper. It is shown that this model may exhibit up to three phase transitions. The specific heat diverges with an exponent $\frac{1}{2}$ either above or below each transition temperature. The exact isotherm for an antiferroelectric model in both staggered and direct fields at a particular temperature is obtained. As the fields vary, the system undergoes transitions among states of zero, partial and complete direct polarization.

1. Introduction

An important progress in the mathematical theory of phase transition is the solution of the ice-rule vertex models and its subsequent development, which have been reviewed by Lieb and Wu (1972). In the past, attention was focused mainly on models with translationally invariant vertex weights. Recently Wu and Lin (1975) considered the staggered ice-rule vertex model on the square lattice which allows different vertex weights for the two sublattices of the square lattice. They pointed out that the staggered ice-rule model is reducible to the Ising model in a non-zero magnetic field and some dimer models of phase transitions (Allen 1974, Salinas and Nagle 1974). In the absence of a general solution, they studied in detail the most general Pfaffian solution. Their results can be summarized as follows. The system may exhibit up to two phase transitions. If there is only one transition temperature T_c, then below T_c the system is in an ordered or frozen state while above T_c the specific heat diverges with an exponent $\alpha = \frac{1}{2}$. If there are two transition temperatures T_c^1 and $T_c^2(> T_c^1)$, then the system is in a frozen state below T_c^1 and in an ordered state above T_c^2 while the specific heat diverges with $\alpha = \frac{1}{2}$ above T_c^1 and $\alpha' = \frac{1}{2}$ below T_c^2. They also obtained the exact isotherm of a general antiferroelectric model at a particular temperature in the presence of both direct and staggered fields. As the fields varied, the system underwent transitions among states of zero, partial and complete direct polarization. The purpose of this paper is to study the most general Pfaffian solution of the staggered ice-rule vertex model on the Kagomé lattice.

2. Definition of the model

Place arrows on the lattice edge of a Kagomé lattice L of N sites subject to the ice-rule that there are always two arrows in and two arrows out at each site. In figure 1, the three sublattices of L are denoted by A, B, and C. The six configurations allowed at each

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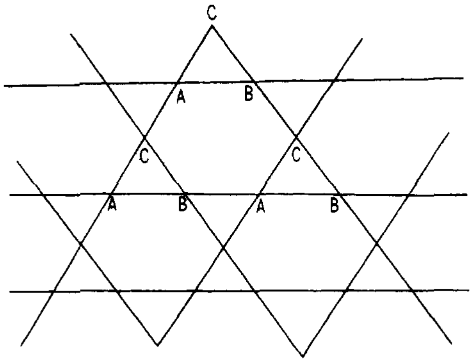

Figure 1. The Kagomé lattice with three sublattices A, B and C.

vertex are shown in figure 2, where each vertex type is assigned a weight. Let the vertex weights be

$$\begin{aligned} \{\omega\} &= \{\omega_1, \omega_2, \ldots, \omega_6\} && \text{on A} \\ \{\omega'\} &= \{\omega'_1, \omega'_2, \ldots, \omega'_6\} && \text{on B} \\ \{\omega''\} &= \{\omega''_1, \omega''_2, \ldots, \omega''_6\} && \text{on C.} \end{aligned} \tag{1}$$

The partition function is

$$Z \equiv \sum (\Pi \omega_i^{n_i})(\Pi \omega_i'^{n_i'})(\Pi \omega_i''^{n_i''}) \tag{2}$$

where the summation is extended to all allowed arrow configurations on L, and $n_i(n'_i, n''_i)$ is the number of the ith-type sites on A(B, C). The goal is to compute the 'free energy'

$$\psi = \lim_{N\to\infty} \frac{1}{N} \ln Z. \tag{3}$$

In a physical model, the vertex weights can be interpreted as the Boltzmann factors

$$\omega_i = \exp(-\beta e_i) \qquad \omega'_i = \exp(-\beta e'_i) \qquad \omega''_i = \exp(-\beta e''_i) \tag{4}$$

where $\beta = 1/kT$ and e_i, e'_i, e''_i are the vertex energies.

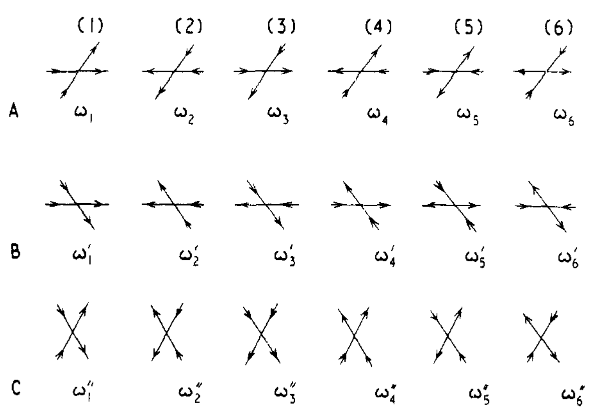

Figure 2. The six ice-rule configurations and the associated vertex weights.

3. Pfaffian solution

A vertex model is soluble by the Pfaffian method (Montroll 1964) if a certain 'free-fermion' condition is satisfied at each vertex (Fan and Wu 1970). For the staggered ice-rule model on the Kagomé lattice, the condition reads

$$\begin{aligned} \omega_1\omega_2+\omega_3\omega_4 &= \omega_5\omega_6 \\ \omega_1'\omega_2'+\omega_3'\omega_4' &= \omega_5'\omega_6' \\ \omega_1''\omega_2''+\omega_3''\omega_4'' &= \omega_5''\omega_6''. \end{aligned} \tag{5}$$

Under this condition the partition function is equal to a Pfaffian which can be evaluated exactly. With the details outlined in appendix 1, the result is given here:

$$\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ln|F(\theta,\phi)| \tag{6}$$

where

$$F = a+b\,e^{i\theta}+b'\,e^{-i\theta}+c\,e^{i\phi}+c'\,e^{-i\phi}-d\,e^{i(\theta+\phi)}-d'\,e^{-i(\theta+\phi)} \tag{7}$$

with

$$\begin{aligned} a &= \omega_3\omega_4'\omega_2''+\omega_4\omega_3'\omega_1''+\omega_5\omega_5'\omega_6''+\omega_6\omega_6'\omega_5'' \\ b &= \omega_4\omega_2'\omega_4'' \qquad b' = \omega_3\omega_1'\omega_3'' \\ c &= \omega_2\omega_3'\omega_3'' \qquad c' = \omega_1\omega_4'\omega_4'' \\ d &= \omega_2\omega_2'\omega_2'' \qquad d' = \omega_1\omega_1'\omega_1''. \end{aligned} \tag{8}$$

Notice that although there are 15 independent vertex weights to start with, the final expression contains only seven independent parameters. The free-fermion condition (5) implies two inequalities (see appendix 2)

$$a \geqslant 2(bb')^{1/2}+2(cc')^{1/2}+2(dd')^{1/2} \tag{9}$$

$$a \geqslant 3(bcd'+b'c'd)^{1/3}. \tag{10}$$

The special case of $d = d' = 0$ has been discussed by Wu and Lin (1975).

The analytic properties of ψ will be discussed in appendix 2, where the following results are proved.

Consider a, b, b', c, c', d, d' as independent and positive parameters which satisfy the inequality (9). We have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\max\{b,b'\} \qquad \text{if } b+b' \geqslant a+c+c'+d+d' \\ &= \tfrac{1}{2}\ln\max\{c,c'\} \qquad \text{if } c+c' \geqslant a+b+b'+d+d' \\ &= \tfrac{1}{2}\ln\max\{d,d'\} \qquad \text{if } d+d' \geqslant a+b+b'+c+c' \\ &= \frac{1}{8\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln[a-2(bb')^{1/2}\cos\theta-2(cc')^{1/2}\cos\phi \\ &\qquad -2(dd')^{1/2}\cos(\theta+\phi)] \\ &\qquad \text{if } a \geqslant b+b'+c+c'+d+d' \text{ and } bcd' = b'c'd. \end{aligned} \tag{11}$$

If a, $b+b'$, $c+c'$, $d+d'$ form a polygon and $bcd' = b'c'd$, then $F(\theta_0, \phi_0) = 0$ has exactly one solution such that $0 < \theta_0(\phi_0) < \pi$ and we have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\max\{b, d\} + \frac{1}{2\pi}\int_0^{\phi_0} \ln|z_1(\phi)|\,\mathrm{d}\phi && \text{if } b'+d' < b+d \\ &= \tfrac{1}{2}\ln\max\{b', d'\} + \frac{1}{2\pi}\int_0^{\phi_0} \ln|z_2(\phi)^{-1}|\,\mathrm{d}\phi && \text{if } b'+d' > b+d \end{aligned} \tag{12}$$

where $z_1, z_2(|z_1| \geqslant |z_2|)$ are the roots of

$$(b - d\,\mathrm{e}^{\mathrm{i}\phi})z^2 + (a + c\,\mathrm{e}^{\mathrm{i}\phi} + c'\,\mathrm{e}^{-\mathrm{i}\phi})z + b' - d'\,\mathrm{e}^{-\mathrm{i}\phi} = 0.$$

In the special case $b = d = 0$, equation (12) reduces to

$$\psi = \tfrac{1}{2}\ln\max\{b', d'\} + \frac{1}{2\pi}\int_0^{\phi_0} \mathrm{d}\phi \ln\left|\frac{a + c\,\mathrm{e}^{\mathrm{i}\phi} + c'\,\mathrm{e}^{-\mathrm{i}\phi}}{b' - d'\,\mathrm{e}^{-\mathrm{i}\phi}}\right| \tag{13}$$

where

$$4cc'\cos^2\phi_0 + 2[b'd' + a(c+c')]\cos\phi_0 + a^2 + (c-c')^2 - b'^2 - d'^2 = 0.$$

When the polygon degenerates into a straight line, namely

$$\Delta \equiv a + b + b' + c + c' + d + d' - 2\max\{a, b+b', c+c', d+d'\} = 0 \tag{14}$$

we have

$$\begin{aligned} &\theta_0 = 0 && \phi_0 = 0 && \text{if } a = b+b'+c+c'+d+d' \\ &\theta_0 = 0 && \phi_0 = \pi && \text{if } b+b' = a+c+c'+d+d' \\ &\theta_0 = \pi && \phi_0 = 0 && \text{if } c+c' = a+b+b'+d+d' \\ &\theta_0 = \pi && \phi_0 = \pi && \text{if } d+d' = a+b+b'+c+c'. \end{aligned} \tag{15}$$

If $bcd' = b'c'd$, then ψ is non-analytic if and only if the parameters satisfy the critical condition (14). Besides, we have

$$\begin{aligned} \psi_{\text{singular}} &\sim \Delta^2 \ln(-\Delta) && \Delta \to 0^- && \text{if } b = b', c = c', d = d' \\ &\sim \Delta^{3/2} && \Delta \to 0^+ && \text{otherwise.} \end{aligned} \tag{16}$$

The physical interpretations of these results are given in the following two sections.

4. Exactly soluble models

In physical models, the vertex weights are the Boltzman factors

$$\omega_i = \exp(-\beta e_i) \qquad \omega_i' = \exp(-\beta e_i') \qquad \omega_i'' = \exp(-\beta e_i'').$$

In this section we consider those models for which the free-fermion condition (5) is to hold for all temperatures so that the model is exactly solved. Notice that ψ is invariant under the following three transformations:

$$\begin{aligned} T_1:&\quad \omega_1(\omega_1', \omega_1'', \omega_3, \omega_3', \omega_3'') \leftrightarrow \omega_2(\omega_2', \omega_2'', \omega_4, \omega_4', \omega_4'') \\ T_2:&\quad \omega_1(\omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_3'') \leftrightarrow \omega_1'(\omega_2', \omega_4', \omega_3', \omega_5', \omega_6', \omega_4'') \\ T_3:&\quad \omega_1(\omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_3') \leftrightarrow \omega_3''(\omega_4'', \omega_1'', \omega_2'', \omega_6'', \omega_5'', \omega_4'). \end{aligned} \tag{17}$$

There are four distinct classes of exactly soluble models (others are related to them by T_2 and T_3):

$$\begin{aligned}
&\text{A}: \quad \omega_1\omega_2 = \omega_1'\omega_2' = \omega_1''\omega_2'' = 0 \qquad \omega_3\omega_4 = \omega_5\omega_6 \qquad \omega_3'\omega_4' = \omega_5'\omega_6' \\
&\qquad\qquad \omega_3''\omega_4'' = \omega_5''\omega_6'' \\
&\text{B}: \quad \omega_1\omega_2 = \omega_1'\omega_2' = \omega_3''\omega_4'' = 0 \qquad \omega_3\omega_4 = \omega_5\omega_6 \qquad \omega_3'\omega_4' = \omega_5'\omega_6' \\
&\qquad\qquad \omega_1''\omega_2'' = \omega_5''\omega_6'' \\
&\text{C}: \quad \omega_3\omega_4 = \omega_3'\omega_4' = \omega_1''\omega_2'' = 0 \qquad \omega_1\omega_2 = \omega_5\omega_6 \qquad \omega_1'\omega_2' = \omega_5'\omega_6' \\
&\qquad\qquad \omega_3''\omega_4'' = \omega_5''\omega_6'' \\
&\text{D}: \quad \omega_3\omega_4 = \omega_3'\omega_4' = \omega_3''\omega_4'' = 0 \qquad \omega_1\omega_2 = \omega_5\omega_6 \qquad \omega_1'\omega_2' = \omega_5'\omega_6' \\
&\qquad\qquad \omega_1''\omega_2'' = \omega_5''\omega_6''.
\end{aligned} \tag{18}$$

Each class has four different cases (the other four cases are related to them by T_1); therefore we have 16 different models to consider:

(1) $\omega_1 = \omega_2' = \omega_3'' = 0$ (class B) $\quad F = a \quad$ (19)

(2) $\omega_3 = \omega_3' = \omega_4'' = 0$ (class D) $\quad F = a - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)} \quad$ (20)

(3) $\omega_1 = \omega_1' = \omega_2'' = 0$ (class A) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} \quad$ (21)

(4) $\omega_1 = \omega_2' = \omega_1'' = 0$ (class A) $\quad F = a + b'\,\mathrm{e}^{-\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} \quad$ (22)

(5) $\omega_1 = \omega_2' = \omega_2'' = 0$ (class A) $\quad F = a + b'\,\mathrm{e}^{-\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} \quad$ (23)

(6) $\omega_1 = \omega_1' = \omega_3'' = 0$ (class B) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \quad$ (24)

(7) $\omega_1 = \omega_1' = \omega_4'' = 0$ (class B) $\quad F = a + c\,\mathrm{e}^{\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \quad$ (25)

(8) $\omega_1 = \omega_2' = \omega_4'' = 0$ (class B) $\quad F = a + b'\,\mathrm{e}^{-\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} \quad$ (26)

(9) $\omega_3 = \omega_4' = \omega_3'' = 0$ (class D) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)} \quad$ (27)

(10) $\omega_3 = \omega_4' = \omega_4'' = 0$ (class D) $\quad F = a + c\,\mathrm{e}^{\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)} \quad$ (28)

(11) $\omega_1 = \omega_1' = \omega_1'' = 0$ (class A) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \quad$ (29)

(12) $\omega_3 = \omega_4' = \omega_1'' = 0$ (class C) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \quad$ (30)

(13) $\omega_3 = \omega_3' = \omega_1'' = 0$ (class C) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c'\,\mathrm{e}^{-\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} \quad$ (31)

(14) $\omega_3 = \omega_3' = \omega_2'' = 0$ (class C) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c'\,\mathrm{e}^{-\mathrm{i}\phi} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)} \quad$ (32)

(15) $\omega_3 = \omega_3' = \omega_3'' = 0$ (class D) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c'\,\mathrm{e}^{-\mathrm{i}\phi} - d\,\mathrm{e}^{\mathrm{i}(\theta+\phi)} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)} \quad$ (33)

(16) $\omega_3 = \omega_4' = \omega_2'' = 0$ (class C) $\quad F = a + b\,\mathrm{e}^{\mathrm{i}\theta} + c\,\mathrm{e}^{\mathrm{i}\phi} - d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)}. \quad$ (34)

The free energies of these models are discussed in the following subsections, where similar models are discussed together.

4.1. *Model (1)*

In this model we have $\psi = \frac{1}{2}\ln a$ and there is no phase transition.

4.2. *Model (2)*

This model is identical to the model (*b*) of Wu and Lin (1975). The system is in a frozen state at all temperatures and

$$\psi = \tfrac{1}{4}\beta \max\{|e_5+e_5'+e_6''-e_6-e_6'-e_5''|, |e_1+e_1'+e_1''-e_2-e_2'-e_2''|\} \\ -\tfrac{1}{4}\beta(e_1+e_1'+e_1''+e_2+e_2'+e_2''). \tag{35}$$

4.3. *Models (3)–(8)*

The free energies of these models can be rewritten in the form†

$$\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \ln(\Omega_1+\Omega_2\,e^{i\alpha}+\Omega_3\,e^{i\beta}). \tag{36}$$

This integral was evaluated by Wu and Lin (1975). These models are similar to the model (*c*) of Wu and Lin except that Ω_1 is the sum of three (models (3)–(5)) or four (models (6)–(8)) Boltzmann factors instead of two. For completeness we write down the result

$$\psi = \begin{cases} \frac{1}{2}\ln\max\{\Omega_1,\Omega_2,\Omega_3\} & \text{if } 2\max\{\Omega_1,\Omega_2,\Omega_3\} \geqslant \Omega_1+\Omega_2+\Omega_3 \quad (37) \\ \frac{1}{2}\ln\Omega_3+\dfrac{1}{4\pi}\displaystyle\int_{-\phi_1}^{\phi_1} d\phi \ln[(\Omega_1+\Omega_2\,e^{-i\phi})/\Omega_3] & \text{otherwise} \quad (38) \end{cases}$$

where

$$|\Omega_1+\Omega_2\,e^{-i\phi_1}| = \Omega_3. \tag{39}$$

The critical condition is

$$2\max\{\Omega_1,\Omega_2,\Omega_3\} = \Omega_1+\Omega_2+\Omega_3. \tag{40}$$

In the model of Wu and Lin the system may exhibit up to two phase transitions. In our models the system has either zero or two phase transitions. To see this, we denote

$$\Omega_1 = \sum_{i=1}^{3\text{ or }4} \exp(-\beta\epsilon_i) \qquad \Omega_2 = \exp(-\beta\epsilon_5) \qquad \Omega_3 = \exp(-\beta\epsilon_6). \tag{41}$$

If ϵ_5 is the lowest energy, then each of the following equations

$$\Omega_2 = \Omega_1+\Omega_3 \qquad \Omega_1 = \Omega_2+\Omega_3 \tag{42}$$

has one solution at $T = T_c^1$, T_c^2 ($> T_c^1$) respectively and we have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\Omega_1 && T \geqslant T_c^2 \\ &= (38) && T_c^2 \geqslant T \geqslant T_c^1 \\ &= \tfrac{1}{2}\ln\Omega_2 && T \leqslant T_c^1. \end{aligned} \tag{43}$$

† Although the integrand is complex, the integral is real.

The system is in a frozen state below T_c^1 and in an ordered state above T_c^2 while the specific heat diverges with an exponent $\alpha = \frac{1}{2}$ above T_c^1 and $\alpha' = \frac{1}{2}$ below T_c^2.

If ϵ_1 is the lowest energy then $\Omega_1 = \Omega_2 + \Omega_3$ has either 0 or 2 solutions (say $T_c^2 > T_c^1$) and

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\Omega_1 && T_c^1 \geqslant T \text{ or } T \geqslant T_c^2 \\ &= (38) && T_c^2 \geqslant T \geqslant T_c^1. \end{aligned} \tag{44}$$

4.4. Models (9) and (10)

The free energies can be rewritten in the form

$$\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \ln(\Omega_1 + \Omega_2 e^{i\alpha} + f e^{i\beta} + g e^{-i\beta}). \tag{45}$$

These models are similar to the modified KDP model in a staggered field (Wu 1971) except that Ω_1 here is the sum of three Boltzmann factors instead of one. Wu's result is

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\Omega_2 && \text{if } \Omega_2 \geqslant \Omega_1 + f + g && (46) \\ &= \tfrac{1}{2}\ln\max\{f, g\} && \text{if } f + g \geqslant \Omega_1 + \Omega_2 && (47) \\ &= \tfrac{1}{2}\ln[\tfrac{1}{2}\Omega_1 + (\tfrac{1}{4}\Omega_1^2 - fg)^{1/2}] && \text{if } \Omega_1 \geqslant \Omega_2 + f + g && (48) \\ &= \tfrac{1}{2}\ln\Omega_2 + \frac{1}{4\pi}\int_{-\phi_1}^{\phi_1} d\phi \ln[(\Omega_1 + f e^{i\phi} + g e^{-i\phi})/\Omega_2] && \text{otherwise} && (49) \end{aligned}$$

where

$$|\Omega_1 + f e^{i\phi_1} + g e^{-i\phi_1}| = \Omega_2. \tag{50}$$

The critical condition is

$$\Omega_1 + \Omega_2 + f + g = 2\max\{\Omega_1, \Omega_2, f + g\}. \tag{51}$$

In Wu's model there is only one transition. In our models the system may exhibit up to two transitions. To see this, we denote

$$\begin{aligned} \Omega_1 &= \exp(-\beta\epsilon_1) + \exp(-\beta\epsilon_5) + \exp(-\beta\epsilon_6) \\ \Omega_2 &= \exp(-\beta\epsilon_2) \\ f + g &= \exp(-\beta\epsilon_3) + \exp(-\beta\epsilon_4) \end{aligned} \tag{52}$$

where $\epsilon_3 + \epsilon_4 = \epsilon_5 + \epsilon_6$ and $\epsilon_1 = \infty$ in Wu's model. There are three possibilities:

(i)
$$\begin{aligned} &\epsilon_2 < \epsilon_i (i \neq 2) && \text{(1 or 2 transitions)} \\ &f + g < \Omega_1 + \Omega_2 && \\ &\Omega_2 = \Omega_1 + f + g && \text{at } T = T_c^1 \\ &\Omega_1 = \Omega_2 + f + g && \text{at } T = T_c^2 (> T_c^1). \end{aligned} \tag{53}$$

T_c^2 may become infinity. We have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\Omega_2 && \text{(frozen state)} && T \leqslant T_c^1 \\ &= (49) && (\alpha = \alpha' = \tfrac{1}{2}) && T_c^2 \geqslant T \geqslant T_c^1 \\ &= (48) && \text{(ordered state)} && T \geqslant T_c^2. \end{aligned} \tag{54}$$

(ii)

$$\begin{aligned} &\epsilon_3 < \epsilon_i (i \neq 3) && \text{(1 or 2 transitions)} \\ &f+g = \Omega_1 + \Omega_2 && \text{at } T = T_c^1 \\ &\Omega_2 < \Omega_1 + f + g && \\ &\Omega_1 = \Omega_2 + f + g && \text{at } T = T_c^2 (> T_c^1). \end{aligned} \tag{55}$$

T_c^2 may become infinity. We have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\max\{f, g\} && \text{(frozen state)} && T \leqslant T_c^1 \\ &= (49) && (\alpha = \alpha' = \tfrac{1}{2}) && T_c^2 \geqslant T \geqslant T_c^1 \\ &= (48) && \text{(ordered state)} && T \geqslant T_c^2. \end{aligned} \tag{56}$$

(iii)

$$\begin{aligned} &\epsilon_1 \text{ or } \epsilon_5 \text{ is the lowest energy} && \text{(0 or 1 transition)} \\ &\Omega_2 < \Omega_1 + f + g && \\ &f+g < \Omega_1 + \Omega_2 && \\ &\Omega_1 = \Omega_2 + f + g && \text{at } T = T_c. \end{aligned} \tag{57}$$

T_c may become infinity. We have

$$\begin{aligned} \psi &= (48) && \text{(ordered state)} && T \leqslant T_c \\ &= (49) && (\alpha = \tfrac{1}{2}) && T \geqslant T_c. \end{aligned} \tag{58}$$

4.5. Models (11)–(14)

The special case of model (11) where $\omega_i = \omega_i' = \omega_i''$, $\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = \omega_4 = \omega_5 = \omega_6 = u$ was considered by Miyazima and Syozi (1968)† and by Wu (1973)‡, and has only one transition temperature determined by $u = \frac{1}{2}$. The free energies of these models can all be expressed in the form

$$\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} \mathrm{d}\alpha \int_{-\pi}^{\pi} \mathrm{d}\beta \ln|\Omega_1 - \Omega_2\, \mathrm{e}^{\mathrm{i}(\alpha+\beta)} + \Omega_3\, \mathrm{e}^{\mathrm{i}\alpha} + \Omega_4\, \mathrm{e}^{\mathrm{i}\beta}|. \tag{59}$$

This integral has been evaluated by Hsue *et al* (1975). The result is

$$\psi = \tfrac{1}{2}\ln\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} \qquad \text{if } \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \leqslant 2\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} \tag{60}$$

$$= \tfrac{1}{2}\ln\max\{\Omega_1, \Omega_4\} + \frac{1}{8\pi}\int_{-\phi_1}^{\phi_1} \mathrm{d}\phi \ln\left(\frac{\Omega_2^2 + \Omega_3^2 + 2\Omega_2\Omega_3\cos\phi}{\Omega_1^2 + \Omega_4^2 - 2\Omega_1\Omega_4\cos\phi}\right) \qquad \text{otherwise} \tag{61}$$

† They made an error in the calculation of the partition function, see Miyazima and Syozi (1973).
‡ The correct partition function was first obtained by Wu. His remark that the specific heat diverges logarithmically both above and below T_c is not correct.

where

$$\cos\phi_1 = (\Omega_1^2+\Omega_4^2-\Omega_2^2-\Omega_3^2)/2(\Omega_1\Omega_4+\Omega_2\Omega_3). \tag{62}$$

The critical condition is

$$\Omega_1+\Omega_2+\Omega_3+\Omega_4 = 2\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}. \tag{63}$$

Our models are similar to the model of Hsue *et al* except that here Ω_1 is the sum of three (model (11)) or two (models (12)–(14)) Boltzmann factors instead of one. In the model (11) the system may exhibit up to two transitions while in the models (12)–(14) the system has either one or three transitions. To see this, we write in model (11)

$$\begin{aligned} \Omega_i &= \exp(-\beta\epsilon_i) \qquad i \neq 1 \\ \Omega_1 &= \exp(-\beta\epsilon_1)+\exp(-\beta\epsilon_5)+\exp(-\beta\epsilon_6). \end{aligned} \tag{64}$$

There are two possibilities:

(i)
$$\begin{aligned} &\epsilon_1 < \epsilon_i (i \neq 1) && \text{0 or 1 transition} \\ &\Omega_1 = \Omega_2+\Omega_3+\Omega_4 && \text{at } T = T_c \\ &\Omega_1+\Omega_2+\Omega_3+\Omega_4 > 2\max\{\Omega_2, \Omega_3, \Omega_4\}. \end{aligned} \tag{65}$$

T_c may become infinity. We have

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\Omega_1 && \text{(ordered state)} && T \leqslant T_c \\ &= (61) && (\alpha = \tfrac{1}{2}) && T \geqslant T_c. \end{aligned} \tag{66}$$

(ii)
$$\begin{aligned} &\min\{\epsilon_2, \epsilon_3, \epsilon_4\} = \text{lowest energy} && \text{(1 or 2 transitions)} \\ &2\max\{\Omega_2, \Omega_3, \Omega_4\} = \Omega_1+\Omega_2+\Omega_3+\Omega_4 && \text{at } T = T_c^1 \\ &\Omega_2+\Omega_3+\Omega_4 = \Omega_1 && \text{at } T = T_c^2 (> T_c^1). \end{aligned} \tag{67}$$

T_c^2 may become infinity. We have

$$\begin{aligned} \psi &= -\tfrac{1}{2}\beta\min\{\epsilon_2, \epsilon_3, \epsilon_4\} && \text{(frozen state)} && T \leqslant T_c^1 \\ &= (61) && (\alpha = \alpha' = \tfrac{1}{2}) && T_c^2 \geqslant T \geqslant T_c^1 \\ &= \tfrac{1}{2}\ln\Omega_1 && \text{(ordered state)} && T \geqslant T_c^2. \end{aligned} \tag{68}$$

In the models (12)–(14) we have $\epsilon_6 = \infty$ and there are two possibilities:

(i)
$$\begin{aligned} &\epsilon_1 < \epsilon_i (i \neq 1) && \text{(1 transition)} \\ &\Omega_1 = \Omega_2+\Omega_3+\Omega_4 && \text{at } T = T_c \\ &\Omega_1+\Omega_2+\Omega_3+\Omega_4 > 2\max\{\Omega_2, \Omega_3, \Omega_4\} \\ &\psi = \tfrac{1}{2}\ln\Omega_1 && \text{(ordered state)} \quad T \leqslant T_c \\ &\quad = (61) && (\alpha = \tfrac{1}{2}) \quad T \geqslant T_c. \end{aligned} \tag{69}$$

(ii)
$$\begin{aligned} &\min\{\epsilon_2, \epsilon_3, \epsilon_4\} = \text{lowest energy} && \text{(1 or 3 transitions)} \\ &2\max\{\Omega_2+\Omega_3+\Omega_4\} = \Omega_1+\Omega_2+\Omega_3+\Omega_4 && \text{at } T = T_c^1. \end{aligned} \tag{70}$$

$\Omega_2+\Omega_3+\Omega_4 = \Omega_1$ has 0 or 2 solutions (at $T = T_c^3$, T_c^2 such that $T_c^3 > T_c^2 > T_c^1$). We have

$$\begin{aligned}\psi &= -\tfrac{1}{2}\beta \min(\epsilon_2, \epsilon_3, \epsilon_4) && \text{(frozen state)} && T \leqslant T_c^1 \\ &= (61) && (\alpha = \alpha' = \tfrac{1}{2}) && T \geqslant T_c^3 \text{ or } T_c^2 \geqslant T \geqslant T_c^1 \\ &= \tfrac{1}{2}\ln \Omega_1 && \text{(ordered state)} && T_c^3 \geqslant T \geqslant T_c^2.\end{aligned} \tag{71}$$

4.6. Model (15)

This model satisfies the condition $bcd' = b'c'd$ and we can use the results of § 3. The system has exactly one transition temperature T_c determined by the critical condition (14). We denote

$$\begin{aligned} &a = a_1+a_2 \qquad a_1 = \omega_5\omega_5'\omega_6'' = \exp(-\beta\epsilon_1) \qquad a_2 = \omega_6\omega_6'\omega_5'' = \exp(-\beta\epsilon_2) \\ &b = \exp(-\beta\epsilon_3) \qquad c' = \exp(-\beta\epsilon_4) \qquad d = \exp(-\beta\epsilon_5) \qquad d' = \exp(-\beta\epsilon_6)\end{aligned} \tag{72}$$

where $\epsilon_1+\epsilon_2 = \epsilon_5+\epsilon_6$. It follows from equations (11) and (13) that

$$\begin{aligned}\psi &= -\tfrac{1}{2}\beta \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\} && \text{(frozen state)} && T \leqslant T_c \\ &= -\tfrac{1}{2}\beta \min\{\epsilon_3, \epsilon_4\} + \frac{1}{2\pi}\int_0^{\phi_0} d\phi \ln\left|\frac{a+d\,e^{i\phi}+d'\,e^{-i\phi}}{b-c'\,e^{-i\phi}}\right| && (\alpha = \tfrac{1}{2}) && T \geqslant T_c\end{aligned} \tag{73}$$

where

$$4dd'\cos^2\phi_0 + 2[bc'+a(d+d')]\cos\phi_0 + a^2+(d-d')^2-b^2-c'^2 = 0. \tag{74}$$

4.7. Model (16)

This is the only model with $bcd' \neq b'c'd$. The analytic property of this model is discussed in appendix 3. The free energy

$$\psi(a,b,c,d') = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ln|a+b\,e^{i\theta}+c\,e^{i\phi}-d'\,e^{-i(\theta+\phi)}| \tag{75}$$

is symmetric in b, c, d'. We have

$$\psi = \tfrac{1}{2}\ln\max\{b,c,d'\} \qquad \text{if } 2\max\{b,c,d'\} \geqslant a+b+c+d' \tag{76}$$

$$= \psi[a,(bcd')^{1/3},(bcd')^{1/3},(bcd')^{1/3}] \qquad \text{if } a \geqslant b+c+d' \tag{77}$$

$$= \left(\frac{1}{2}-\frac{\phi_0}{2\pi}\right)\ln\max\{b,d'\} + \frac{1}{2\pi}\int_0^{\phi_0} d\phi \ln \tfrac{1}{2}[y^{1/2}+(y+4bd')^{1/2}] \qquad \text{otherwise} \tag{78}$$

where y is the positive root of

$$\begin{aligned} f(y,\cos\phi) &= y^2-(a^2+c^2+2ac\cos\phi-4bd')-2bd'(1+\cos\phi)(2c\cos\phi+a-c)^2 \\ &= 0\end{aligned} \tag{79}$$

and $f((b-d)^2, \cos\phi_0) = 0$. The critical condition is

$$\Delta \equiv a+b+c+d'-2\max\{a,b,c,d'\} = 0 \tag{80}$$

and we have

$$\psi_{\text{singular}} \sim \Delta^{3/2} \qquad \Delta \to 0^+. \tag{81}$$

The system may exhibit up to two phase transitions. To see this, we denote

$$
\begin{aligned}
a &= \exp(-\beta\epsilon_1)+\exp(-\beta\epsilon_2)+\exp(-\beta\epsilon_3) \\
b &= \exp(-\beta\epsilon_4) \qquad c = \exp(-\beta\epsilon_5) \qquad d' = \exp(-\beta\epsilon_6)
\end{aligned}
\tag{82}
$$

where $\epsilon_1+\epsilon_2+\epsilon_3 = \epsilon_4+\epsilon_5+\epsilon_6$. There are two possibilities:

(i) $\min\{\epsilon_4, \epsilon_5, \epsilon_6\}$ = lowest energy (1 or 2 transitions)

$2\max\{b, c, d'\} = a+b+c+d'$ at $T = T_c^1$

$a = b+c+d'$ at $T = T_c^2(> T_c^1)$.

T_c^2 may become infinity. We have

$$
\begin{aligned}
\psi &= -\tfrac{1}{2}\beta\min\{\epsilon_4, \epsilon_5, \epsilon_6\} && \text{(frozen state)} && T \leqslant T_c^1 \\
&= (78) && (\alpha = \alpha' = \tfrac{1}{2}) && T_c^2 \geqslant T \geqslant T_c^1 \\
&= (77) && \text{(ordered state)} && T \geqslant T_c^2.
\end{aligned}
\tag{83}
$$

(ii) $\min(\epsilon_1, \epsilon_2, \epsilon_3)$ = lowest energy (0 or 1 transition)

$a = b+c+d'$ at $T = T_c$.

T_c may become infinity. We have

$$
\begin{aligned}
\psi &= (77) && \text{(ordered state)} && T \leqslant T_c \\
&= (78) && (\alpha = \tfrac{1}{2}) && T \geqslant T_c.
\end{aligned}
\tag{84}
$$

5. Exact isotherm of an antiferroelectric model

Following Baxter (1970), we use the free-fermion condition (5) to define a temperature at which the Pfaffian solution is valid. Since the validity of condition (5) is independent of the direct and staggered fields, we have an exact isotherm for a general staggered model.

We denote the staggered field by s, the direct fields in the horizontal and vertical directions by h and v such that

$$
\begin{aligned}
&e_5 = e_5' = e_6'' = s && e_6 = e_6' = e_5'' = -s \\
&e_1 = \epsilon+\tfrac{3}{2}h+\tfrac{1}{2}v && e_2 = \epsilon-\tfrac{3}{2}h-\tfrac{1}{2}v && e_3 = \epsilon'+\tfrac{1}{2}h-\tfrac{1}{2}v && e_4 = \epsilon'-\tfrac{1}{2}h+\tfrac{1}{2}v \\
&e_1' = \epsilon+\tfrac{3}{2}h-\tfrac{1}{2}v && e_2' = \epsilon-\tfrac{3}{2}h+\tfrac{1}{2}v && e_3' = \epsilon'-\tfrac{1}{2}h-\tfrac{1}{2}v && e_4' = \epsilon'+\tfrac{1}{2}h+\tfrac{1}{2}v \\
&e_1'' = \epsilon'(\epsilon)+h && e_2'' = \epsilon'(\epsilon)-h && e_3'' = \epsilon(\epsilon')-v && e_4'' = \epsilon(\epsilon')+v.
\end{aligned}
\tag{85}
$$

The temperature is determined by equation (5)

$$
e^{-2\beta\epsilon}+e^{-2\beta\epsilon'} = 1. \tag{86}
$$

Note that $bcd' = b'c'd$ and the results of § 3 can be used.

5.1. The case with $e_1+e_2 = e_1'+e_2' = e_3''+e_4''$

The free energy is

$$
\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ln|F(\theta, \phi)| \tag{87}
$$

where

$$F = 2\cosh S + 2\,\mathrm{e}^{-3\beta\epsilon'} + 2\,\mathrm{e}^{-\beta(\epsilon'+2\epsilon)}[\cosh(H-V+\mathrm{i}\theta) + \cosh(H+V+\mathrm{i}\phi) - \cosh(2H+\mathrm{i}\theta+\mathrm{i}\phi)] \tag{88}$$

$$S = 3\beta s \qquad H = 2\beta h \qquad V = 2\beta v.$$

We define

$$g(S) = \mathrm{e}^{\beta(\epsilon'+2\epsilon)}\cosh S + \mathrm{e}^{2\beta(\epsilon-\epsilon')}.$$

It follows from equations (11) and (12) that

$$\psi = \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \ln[g(S) - \cos\theta - \cos\phi - \cos(\theta+\phi)] + \tfrac{1}{2}\ln 2 - \beta(\epsilon + \tfrac{1}{2}\epsilon')$$

$$\text{if } g(S) \geqslant \cosh(H+V) + \cosh(H-V) + \cosh(2H) \tag{89}$$

where ψ is independent of H and V, and both direct polarizations P_H and P_V are zero.

$$\psi = |H| - \beta(\epsilon + \tfrac{1}{2}\epsilon') \qquad \text{if } \cosh(2H) \geqslant \cosh(H-V) + \cosh(H+V) + g(S) \tag{90}$$

where

$$P_H = \pm 1 \qquad P_V = P_S = 0$$

$$\psi = \tfrac{1}{2}(|H| + |V|) - \beta(\epsilon + \tfrac{1}{2}\epsilon') \qquad \text{if } \cosh(|H| + |V|) \geqslant \cosh(|H| - |V|) + \cosh(2H) + g(S) \tag{91}$$

where

$$|P_H| = |P_V| = \tfrac{1}{2} \qquad P_S = 0$$

$$\psi = (12) \qquad \text{if } g(S), \cosh(H+V), \cosh(H-V), \cosh(2H) \text{ form a polygon} \tag{92}$$

where†

$$P_H = 1 - \frac{1}{\pi}(\phi_0 + \theta_0) \qquad P_V = 1 - \frac{1}{\pi}(\phi_0 - \theta_0).$$

5.2. *The case with $e_1 + e_2 = e_1' + e_2' = e_1'' + e_2''$*

In this case we have

$$F = 2\cosh S + 2\,\mathrm{e}^{-\beta(2\epsilon'+\epsilon)}[1 + \cosh(H-V+\mathrm{i}\theta) + \cosh(H+V+\mathrm{i}\phi)] - 2\,\mathrm{e}^{-3\beta\epsilon}\cosh(2H+\mathrm{i}\theta+\mathrm{i}\phi). \tag{93}$$

We define

$$f(S) = 1 + \mathrm{e}^{\beta(2\epsilon'+\epsilon)}\cosh S. \tag{94}$$

It follows from equations (11) and (12) that

$$\psi = \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \ln[f(S) - \cos\theta - \cos\phi - \mathrm{e}^{2\beta(\epsilon'-\epsilon)}\cos(\theta+\phi)] + \tfrac{1}{2}\ln 2 - \beta(\epsilon' + \tfrac{1}{2}\epsilon)$$

$$\text{if } f(S) \geqslant \cosh(H-V) + \cosh(H+V) + \mathrm{e}^{2\beta(\epsilon'-\epsilon)}\cosh(2H) \tag{95}$$

† See equation (65) of Wu and Lin (1975).

where $P_H = P_V = 0$.

$$\psi = |H| - \tfrac{3}{2}\beta\epsilon \qquad \text{if } e^{2\beta(\epsilon'-\epsilon)}\cosh(2H) \geqslant \cosh(H-V)+\cosh(H+V)+f(S) \tag{96}$$

where $P_H = \pm 1$, $P_V = P_S = 0$.

$$\psi = \tfrac{1}{2}(|H|+|V|) - \beta(\epsilon' + \tfrac{1}{2}\epsilon)$$

$$\text{if } \cosh(|H|+|V|) \geqslant \cosh(|H|-|V|) + e^{2\beta(\epsilon'-\epsilon)}\cosh(2H) + f(S) \tag{97}$$

where $|P_H| = |P_V| = \frac{1}{2}$, $P_S = 0$.

$$\psi = (12) \qquad \text{otherwise} \tag{98}$$

where

$$P_H = 1 - \frac{1}{\pi}(\phi_0 + \theta_0) \qquad P_V = 1 - \frac{1}{\pi}(\phi_0 - \theta_0).$$

6. Conclusion

We have considered all soluble models, where the vertex weights satisfy the free-fermion condition at all temperatures. We found that the system may exhibit up to three phase transitions. If there is only one transition, then the system is in an ordered or frozen state below T_c, and the specific heat diverges with $\alpha = \frac{1}{2}$ above T_c. If there are two transitions ($T_c^2 > T_c^1$), then the system is in an ordered state above T_c^2, and in an ordered (in this case the free energy is described by the same function for $T > T_c^2$ and $T < T_c^1$) or frozen state below T_c^1, while the specific heat diverges with $\alpha = \frac{1}{2}$ above T_c^1 and $\alpha' = \frac{1}{2}$ below T_c^2. If there are three transitions ($T_c^3 > T_c^2 > T_c^1$), then the system is frozen below T_c^1, and in an ordered state for $T_c^3 \geqslant T \geqslant T_c^2$, while the specific heat diverges with $\alpha = \frac{1}{2}$ above T_c^3, T_c^1 and $\alpha' = \frac{1}{2}$ below T_c^2. In the last case, the free energy is described by the same function for both $T > T_c^3$ and $T_c^2 > T > T_c^1$.

We also obtained the exact isotherm of a general antiferroelectric model at a particular temperature in the presence of both direct and staggered fields. As the fields varied, the system underwent transitions among states of zero direct polarizations ($P_H = P_V = 0$), partial polarizations and complete direct polarizations ($P_S = 0$, $|P_H| + |P_V| = 1$).

Acknowledgment

The author is indebted to Professor F Y Wu for suggesting this problem and encouragement.

Appendix 1. Pfaffian solution

Expand each site of L into a 'city' of four terminals to form a dimer lattice L^Δ whose unit cell is shown in figure 3. Following exactly the same procedure of Wu and Lin (1975), we obtain

$$\psi = \frac{1}{4(2\pi)^2}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \, \ln[(\omega_2\omega_2'\omega_2'')^2 D(\theta,\phi)] \tag{A.1}$$

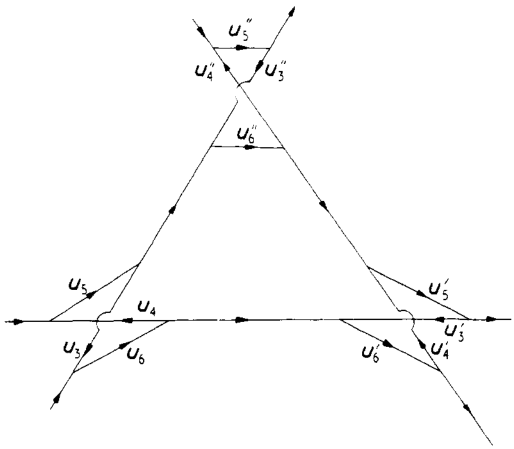

Figure 3. A unit cell of the dimer lattice L^{Δ}.

where

$$D(\theta,\phi)=\begin{vmatrix}0 & \mathbf{A}\\ -\mathbf{A}^* & 0\end{vmatrix}$$

$$\mathbf{A}=\begin{vmatrix} u_3 & -u_6 & 0 & 0 & 0 & 1\\ -u_5 & u_4 & 0 & 1 & 0 & 0\\ 0 & 0 & u_3'' & -u_6'' & 1 & 0\\ 0 & e^{i\phi} & -u_5'' & u_4'' & 0 & 0\\ 0 & 0 & e^{i\theta} & 0 & u_3' & -u_6'\\ e^{i(\theta+\phi)} & 0 & 0 & 0 & -u_5' & u_4' \end{vmatrix} \tag{A.2}$$

$$u_i=\omega_i/\omega_2 \qquad u_i'=\omega_i'/\omega_2' \qquad u_i''=\omega_i''/\omega_2''$$

and $\mathbf{A}^*$ is the hermitian conjugate matrix of $\mathbf{A}$. Equation (A.1) reduces to equation (6) in the text after some algebra.

Appendix 2. General properties of ψ

In this appendix we discuss the analytic properties of the free energy

$$\psi(a,b,b',c,c',d,d')$$
$$=\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{-\pi}^{\pi}\mathrm{d}\phi\,\ln|F(\theta,\phi)| = \mathrm{Re}\,\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{-\pi}^{\pi}\mathrm{d}\phi\,\ln F(\theta,\phi) \tag{A.3}$$

where

$$F=a+b\,e^{i\phi}+b'\,e^{-i\theta}+c\,e^{i\phi}+c'\,e^{-i\phi}-d\,e^{i(\theta+\phi)}-d'\,e^{-i(\theta+\phi)}$$

and the parameters are defined by equation (8).

Lemma 1.

$$a \geqslant 2(bb')^{1/2} + 2(cc')^{1/2} + 2(dd')^{1/2}. \tag{A.4}$$

Proof. The free-fermion condition (5) implies

$$\omega_1\omega_2 = \sin^2\alpha\,\omega_5\omega_6 \qquad \omega_3\omega_4 = \cos^2\alpha\,\omega_5\omega_6$$
$$\omega'_1\omega'_2 = \sin^2\beta\,\omega'_5\omega'_6 \qquad \omega'_3\omega'_4 = \cos^2\beta\,\omega'_5\omega'_6$$
$$\omega''_1\omega''_2 = \cos^2\gamma\,\omega''_5\omega''_6 \qquad \omega''_3\omega''_4 = \sin^2\gamma\,\omega''_5\omega''_6.$$

We rewrite both sides of (A.4) in the form

$$\begin{aligned}
a &= \omega_3\omega'_4\omega''_2 + \omega_4\omega'_3\omega''_1 + \omega_5\omega'_5\omega''_6 + \omega_6\omega'_6\omega''_5 \\
&\geqslant 2(\omega_3\omega_4\omega'_3\omega'_4\omega''_1\omega''_2)^{1/2} + 2(\omega_5\omega_6\omega'_5\omega'_6\omega''_5\omega''_6)^{1/2} \\
&= 2(\omega_5\omega_6\omega'_5\omega'_6\omega''_5\omega''_6)^{1/2}(1 + \cos\alpha\cos\beta\cos\gamma)
\end{aligned}$$

$$\begin{aligned}
&2(bb')^{1/2} + 2(cc')^{1/2} + 2(dd')^{1/2} \\
&\quad = 2(\omega_5\omega_6\omega'_5\omega'_6\omega''_5\omega''_6)^{1/2}(\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\beta\sin\gamma \\
&\qquad + \sin\alpha\sin\beta\cos\gamma).
\end{aligned}$$

The inequality then follows from

$$\begin{aligned}
&1 + \cos(\alpha + \beta + \gamma) \\
&\quad = 1 + \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\
&\qquad - \sin\alpha\cos\beta\sin\gamma \geqslant 0.
\end{aligned}$$

Lemma 2.

$$a \geqslant 3(bcd' + b'c'd)^{1/3}. \tag{A.5}$$

Proof.

$$\begin{aligned}
a &= (\omega_4\omega'_3\omega''_1 + \omega_3\omega'_4\omega''_2) + \omega_5\omega'_5\omega''_6 + \omega_6\omega'_6\omega''_5 \\
&\geqslant 3[(\omega_4\omega'_3\omega''_1 + \omega_3\omega'_4\omega''_2)\omega_5\omega'_5\omega''_6\omega_6\omega'_6\omega''_5]^{1/3} \\
&\geqslant 3[(\omega_4\omega'_3\omega''_1 + \omega_3\omega'_4\omega''_2)\omega_1\omega_2\omega'_1\omega'_2\omega''_3\omega''_4]^{1/3} \\
&= 3(bcd' + b'c'd)^{1/3}.
\end{aligned}$$

From now on we consider a, b, b', c, c', d, d' as independent and positive parameters which satisfy inequalities (A.4) and (A.5).

Lemma 3.

$$\begin{aligned}
&\psi(a, b, b', c, c', d, d') \\
&\quad = \psi(a, b', b, c', c, d', d) = \psi(a, c, c', b, b', d, d') \\
&\quad = \psi(a, d, d', c', c, b, b') = \psi(a, c', c, d, d', b, b').
\end{aligned} \tag{A.6}$$

Proof. These equalities follow from the fact that the integral (A.3) remains the same if $F(\theta, \phi)$ is replaced by each of

$$F(-\theta, -\phi) \qquad F(\phi, \theta) \qquad F(\theta + \phi + \pi, -\phi) \qquad F(-\theta - \phi - \pi, \phi).$$

To calculate ψ we need the following mathematical lemmas (Wu and Lin 1975).

Lemma 4. For complex A, B

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \ln(A\,\mathrm{e}^{\mathrm{i}\theta}+B) = \begin{cases} 2\pi \ln A & \text{if } |A| \geqslant |B| \\ 2\pi \ln B & \text{if } |A| \leqslant |B|. \end{cases} \tag{A.7}$$

Lemma 5. For complex A, B, C

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \ln(A\,\mathrm{e}^{\mathrm{i}\theta}+B+C\,\mathrm{e}^{-\mathrm{i}\theta}) = \begin{cases} 2\pi \ln C & \text{if } |z_1|, |z_2| \geqslant 1 \\ 2\pi \ln A & \text{if } |z_1|, |z_2| \leqslant 1 \\ 2\pi \ln(-Az_1) & \text{if } |z_1| \geqslant 1 \geqslant |z_2| \end{cases} \tag{A.8}$$

where z_1, z_2 are the two roots of the quadratic equation

$$Az^2+Bz+C = 0.$$

Theorem 1. If a, $b+b'$, $c+c'$ and $d+d'$ cannot form a polygon, then $F(\theta, \phi) \neq 0$ for all θ and ϕ which implies ψ has no singularity. In this case we have

$$\begin{aligned} \psi &= \tfrac{1}{2} \ln \max\{b, b'\} && \text{if } b+b' > a+c+c'+d+d' \\ &= \tfrac{1}{2} \ln \max\{c, c'\} && \text{if } c+c' > a+b+b'+d+d' \\ &= \tfrac{1}{2} \ln \max\{d, d'\} && \text{if } d+d' > a+b+b'+c+c' \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{d}\phi \ln x_1(\phi) && \text{if } a > b+b'+c+c'+d+d' \end{aligned} \tag{A.9}$$

where $|x_1| \geqslant |x_2|$ and $x_{1,2}$ satisfy $x^2-Bx+AC = 0$ with

$$A = b-d\,\mathrm{e}^{\mathrm{i}\phi} \qquad B = a+c\,\mathrm{e}^{\mathrm{i}\phi}+c'\,\mathrm{e}^{-\mathrm{i}\phi} \qquad C = b'-d'\,\mathrm{e}^{-\mathrm{i}\phi}. \tag{A.10}$$

Proof. To prove $F(\theta, \phi) \neq 0$, it is sufficient to consider the case

$$b+b' > a+c+c'+d+d' \tag{A.11}$$

since the other cases are either trivial or equivalent to this one by (A.6). Lemma 1 implies

$$a = \epsilon^2+2(bb')^{1/2}+2(cc')^{1/2}+2(dd')^{1/2}.$$

We rewrite (A.11) in the form

$$(\sqrt{b}-\sqrt{b'})^2 > \epsilon^2+(\sqrt{c}+\sqrt{c'})^2+(\sqrt{d}+\sqrt{d'})^2. \tag{A.12}$$

If $F(\theta, \phi) = 0$, then

$$b\,\mathrm{e}^{\mathrm{i}(\theta+\pi)}+b'\,\mathrm{e}^{-\mathrm{i}(\theta+\pi)} = a+c\,\mathrm{e}^{\mathrm{i}\phi}+c'\,\mathrm{e}^{-\mathrm{i}\phi}+d\,\mathrm{e}^{\mathrm{i}(\theta+\phi+\pi)}+d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi+\pi)}$$

or

$$\begin{aligned} &\{\sqrt{b}\exp[\tfrac{1}{2}\mathrm{i}(\theta+\pi)]-\sqrt{b'}\exp[-\tfrac{1}{2}\mathrm{i}(\theta+\pi)]\}^2 \\ &\quad = \epsilon^2+[\sqrt{c}\exp(\tfrac{1}{2}\mathrm{i}\phi)+\sqrt{c'}\exp(-\tfrac{1}{2}\mathrm{i}\phi)]^2 \\ &\quad\quad +\{\sqrt{d}\exp[\tfrac{1}{2}\mathrm{i}(\theta+\phi+\pi)]+\sqrt{d'}\exp[-\tfrac{1}{2}\mathrm{i}(\theta+\phi+\pi)]\}^2. \end{aligned} \tag{A.13}$$

Equation (A.13) is impossible to satisfy for any θ and ϕ since

$$|\text{left hand side of (A.13)}|^2$$
$$\geqslant (\sqrt{b}-\sqrt{b'})^2 > \epsilon^2+(\sqrt{c}+\sqrt{c'})^2+(\sqrt{d}+\sqrt{d'})^2$$
$$\geqslant |\text{right hand side of (A.13)}|^2.$$

To evaluate ψ, we use lemma 5 to carry out the θ integration. Let $z_1(\phi)$ and $z_2(\phi)$ be the two roots of $Az^2+Bz+C = 0$. Since $F(\theta, \phi) = A\,\mathrm{e}^{\mathrm{i}\theta}+B+C\,\mathrm{e}^{-\mathrm{i}\theta} \neq 0$ for all θ, ϕ, it is clear $|z_{1,2}(\phi)| \neq 1$ for all ϕ. Lemma 3 implies that all we need to do is to check the following two cases

(i) $b > b'$, $b+b' > a+c+c'+d+d'$
(ii) $a > b+b'+c+c'+d+d'$.

In the first case, we find $|z_{1,2}(\phi)| < 1$ for all ϕ and

$$\psi = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, \ln(b-d\,\mathrm{e}^{\mathrm{i}\phi}) = \tfrac{1}{2}\ln b.$$

In the second case, we find $|z_1(\phi)| > 1 > |z_2(\phi)|$ for all ϕ and

$$\psi = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, \ln x_1(\phi) \qquad (x_1 \equiv -Az_1).$$

Theorem 2. If $bcd' = b'c'd$ and $a > b+b'+c+c'+d+d'$, then

$$\psi = \frac{1}{8\pi^2}\int_{-\pi}^{\pi} \mathrm{d}\theta \int_{-\pi}^{\pi} \mathrm{d}\phi\, \ln[a+2(bb')^{1/2}\cos\theta+2(cc')^{1/2}\cos\phi-2(dd')^{1/2}\cos(\theta+\phi)]. \tag{A.14}$$

Proof. From theorem 1 we have

$$\psi = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, \ln x_1(\phi)$$

where x_1 satisfies $x^2-Bx+AC = 0$. In the complex $\mathrm{e}^{\mathrm{i}\phi}$ plane the integrand has square root branch points at the four roots of $B^2-4AC = 0$. Defining $c/c' = \mathrm{e}^{2h}$, $(bb')^{1/2} = b_0$, $(cc')^{1/2} = c_0$, $(dd')^{1/2} = d_0$, $y = 2\cosh(h+\mathrm{i}\phi)$, we have

$$f(\mathrm{e}^{\mathrm{i}\phi}) = B^2-4AC = c_0^2y^2+2(ac_0+2b_0d_0)y+a^2-4(b_0^2+d_0^2) = 0. \tag{A.15}$$

It can be established from (A.15) that all four roots of $f(\mathrm{e}^{\mathrm{i}\phi}) = 0$ lie on the negative real axis. Furthermore, the inequality $a > b+b'+c+c'+d+d'$ ensures that two of these roots lie outside the unit circle and two inside the unit circle such that on the real axis these two pairs are separated by the interval $(-\mathrm{e}^{-h}, -1)$. We may then move the contour of the ϕ integration from $-\pi \to \pi$ to $-\pi+\mathrm{i}h \to \pi+\mathrm{i}h$ and obtain (A.14).

Theorem 3. If a, $b+b'$, $c+c'$, $d+d'$ form a polygon, then $F(\theta, \phi) = 0$ has at least one (but no more than three) solution such that $0 \leqslant \theta, \phi \leqslant \pi$. In the following special cases there exists exactly one solution:

(i) $bcd' = b'c'd$
(ii) $b = c = d' = 0$ or $b' = c' = d = 0$.

When the polygon degenerates into a straight line, namely

$$a+b+b'+c+c'+d+d' = 2\max\{a, b+b', c+c', d+d'\} \tag{A.16}$$

$F(\theta, \phi) = 0$ if and only if

$$\begin{array}{lll} \theta = \phi = 0 & & \text{if } a = b+b'+c+c'+d+d' \\ \theta = 0, & \phi = \pi & \text{if } b+b' = a+c+c'+d+d' \\ \theta = \pi, & \phi = 0 & \text{if } c+c' = a+b+b'+d+d' \\ \theta = \phi = \pi & & \text{if } d+d' = a+b+b'+c+c'. \end{array}$$

Proof. $F(\theta, \phi) = 0$ implies

$$\begin{aligned} a+(c+c')\cos\phi+(b+b')\cos\theta &= (d+d')\cos(\theta+\phi) \\ (c-c')\sin\phi+(b-b')\sin\theta &= (d-d')\sin(\theta+\phi). \end{aligned} \tag{A.17}$$

After some algebra we find

$$f(\cos\phi) = A\cos^3\phi+B\cos^2\phi+C\cos\phi+D = 0 \tag{A.18}$$

where

$$\begin{aligned} A &= 8(bc-b'c')(cd'-c'd) \\ B &= 4a[(b-b')(cd'-c'd)-(d-d')(bc-b'c')] \\ &\quad + 4(bc-b'c')(bc'-b'c)+4(cd-c'd')(c'd-cd')-4(bd-b'd')^2 \\ 2(A+C) &= f(1)-f(-1) \\ 2(B+D) &= f(1)+f(-1) \\ f(1) &= (b-b'-d+d')^2[(a+c+c')^2-(b+b'-d-d')^2] \\ f(-1) &= (b-b'+d-d')^2[(a-c-c')^2-(b+b'+d+d')^2]. \end{aligned}$$

Equations (A.18) has at most 3 solutions.

If a, $b+b'$, $c+c'$, $d+d'$ form a polygon we have

$$f(1) \geqslant 0 \geqslant f(-1)$$

which implies that $f(\cos\phi) = 0$ has at least one solution (real ϕ). In the first special case, we introduce two parameters u and v such that

$$\begin{array}{llll} c = c_0\,\mathrm{e}^{u} & c' = c_0\,\mathrm{e}^{-u} & b = b_0\,\mathrm{e}^{-v} & b' = b_0\,\mathrm{e}^{v} \\ & d = d_0\,\mathrm{e}^{u-v} & d' = d_0\,\mathrm{e}^{v-u}. & \end{array} \tag{A.19}$$

We rewrite A and B:

$$A = 16b_0c_0^2d_0[\cosh u-\cosh(u-2v)] \tag{A.20}$$

$$\begin{aligned} \tfrac{1}{8}B = &-ab_0c_0d_0[\cosh(2v)+\cosh(2u-2v)-2]-b_0^2c_0^2[\cosh(2u)-\cosh(2v)] \\ &- b_0^2d_0^2[\cosh(2u-4v)-1]-c_0^2d_0^2[\cosh(2u)-\cosh(2u-2v)]. \end{aligned} \tag{A.21}$$

Without loss of generality we assume $u > 0$ and $c_0 \leqslant b_0, d_0$. If $f(x)$ has three solutions in the range $|x| \leqslant 1$, then we must have $A > 0$ which implies $u > v > 0$. Besides, we have $(\mathrm{d}^2/\mathrm{d}x^2)f(x) = 0$ at $x = x_0$ where $|x_0| < 1$. However, it can be established from $a > 2(b_0+c_0+d_0)$ that $|B| > 3A$ which implies $|x_0| > 1$. In the second case where $b' = c' = d = 0$, we have

$$A = 8bc^2d' \qquad B = 8abcd'. \tag{A.22}$$

Without loss of generality we assume $c \leqslant b, d$. It can be shown from $a > 3(bcd')^{1/3}$ that $B > 3A$ which implies $|x_0| > 1$.

Theorem 4. If $a, b+b', c+c', d+d'$ form a polygon and $bcd' = b'c'd$, then

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln\max\{b,d\} + \frac{1}{2\pi}\int_0^{\phi_0} \ln|z_1|\,\mathrm{d}\phi && \text{if } b'+d' < b+d \\ &= \tfrac{1}{2}\ln\max\{b',d'\} - \frac{1}{2\pi}\int_0^{\phi_0} \ln|z_2|\,\mathrm{d}\phi && \text{if } b'+d' > b+d \end{aligned} \tag{A.23}$$

where $F(\theta_0, \phi_0) = 0$, $0 < \phi_0 < \pi$, z_1 and z_2 ($|z_1| \geqslant |z_2|$) are the roots of

$$(b - d\,\mathrm{e}^{\mathrm{i}\phi})z^2 + (a + c\,\mathrm{e}^{\mathrm{i}\phi} + c'\,\mathrm{e}^{-\mathrm{i}\phi})z + b' - d'\,\mathrm{e}^{-\mathrm{i}\phi} = 0. \tag{A.24}$$

Proof. We write

$$F(\theta,\phi) = A(\phi)\,\mathrm{e}^{\mathrm{i}\theta} + B(\phi) + C(\phi)\,\mathrm{e}^{-\mathrm{i}\theta}$$

and let $z_{1,2}(\phi)$ be the roots of $Az^2 + Bz + C = 0$. It is easy to show that

$$\begin{aligned} &|z_1| > 1 > |z_2| && \text{if } \phi_0 > \phi \\ &|z_1|, |z_2| > 1 && \text{if } \phi_0 < \phi \text{ and } b'+d' > b+d \\ &|z_1|, |z_2| < 1 && \text{if } \phi_0 < \phi \text{ and } b'+d' < b+d. \end{aligned} \tag{A.25}$$

It follows from lemma 5 that

$$\begin{aligned} \psi &= \frac{1}{2\pi}\int_0^{\phi_0} \ln|Az_1|\,\mathrm{d}\phi + \frac{1}{2\pi}\int_{\phi_0}^{\pi} \ln\max\{b,d\}\,\mathrm{d}\phi && \text{if } b'+d' < b+d \\ &= \frac{1}{2\pi}\int_0^{\phi_0} \ln|C/z_2|\,\mathrm{d}\phi + \frac{1}{2\pi}\int_{\phi_0}^{\pi} \ln\max\{b',d'\}\,\mathrm{d}\phi && \text{if } b'+d' > b+d \end{aligned}$$

which reduces to (A.23).

Equation (A.23) implies that the singular part of ψ behaves as $\Delta^{3/2}$ as $\Delta \to 0^+$ where

$$\Delta = a+b+b'+c+c'+d+d' - 2\max\{a, b+b', c+c', d+d'\}. \tag{A.26}$$

Appendix 3. Analytic property of model (16)

The free energy is

$$\psi(a,b,c,d') = \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{-\pi}^{\pi}\mathrm{d}\phi\,\ln|F(\theta,\phi)| \tag{A.27}$$

where

$$F = a+b\,\mathrm{e}^{\mathrm{i}\theta}+c\,\mathrm{e}^{\mathrm{i}\phi}-d'\,\mathrm{e}^{-\mathrm{i}(\theta+\phi)}.$$

Note that ψ is symmetric in b, c, d'. From (A.9) we have

$$\psi = \tfrac{1}{2}\ln\max\{b, c, d'\} \qquad \text{if } 2\max\{b, c, d'\} \geqslant a+b+c+d'. \tag{A.28}$$

Theorem 5. If $a > b+c+d'$, then

$$\psi = \psi[a, (bcd')^{1/3}, (bcd')^{1/3}, (bcd')^{1/3}]. \tag{A.29}$$

Proof. Let $z_1(\phi)$, $z_2(\phi)$ ($|z_1| \geqslant |z_2|$) be the roots of

$$bz^2+(a+c\,\mathrm{e}^{\mathrm{i}\phi})z-d'\,\mathrm{e}^{-\mathrm{i}\phi} = 0. \tag{A.30}$$

It follows from $a > b+c+d'$ that $|z_1| > 1 > |z_2|$. Lemma 5 of appendix 2 implies

$$\psi = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\,\ln(-bz_1) = \psi[a, (bd')^{1/2}, c, (bd')^{1/2}]. \tag{A.31}$$

The last step follows from the fact that $x = -bz$ satisfies

$$x^2-(a+c\,\mathrm{e}^{\mathrm{i}\phi})x-bd'\,\mathrm{e}^{-\mathrm{i}\phi} = 0. \tag{A.32}$$

The symmetry property of ψ implies

$$\begin{aligned}\psi &= \psi[a, c, (bd')^{1/2}, (bd')^{1/2}] = \psi[a, (c^2bd')^{1/4}, (bd')^{1/2}, (c^2bd')^{1/4}]\\ &= \ldots = \psi[a, (bcd')^{1/3}, (bcd')^{1/3}, (bcd')^{1/3}].\end{aligned}$$

Theorem 6. If a, b, c, d' form a polygon,

$$\psi = \left(\frac{1}{2}-\frac{\phi_0}{2\pi}\right)\ln\max\{b, d'\}+\frac{1}{2\pi}\int_0^{\phi_0}\mathrm{d}\phi\,\ln\tfrac{1}{2}[y^{1/2}+(y+4bd')^{1/2}] \tag{A.33}$$

where y is the positive root of

$$f(y\cos\phi) = y^2-(a^2+c^2+2ac\cos\phi-4bd')-2bd'(1+\cos\phi)(2c\cos\phi+a-c)^2 = 0. \tag{A.34}$$

Proof. Let $z = \rho\,\mathrm{e}^{\mathrm{i}\alpha}$ (α real) be the root of equation (A.30). After some algebra, we find $f(y, \cos\phi) = 0$ where $y = (b\rho-d'\rho^{-1})^2$. If a, b, c, d' form a polygon, $F(\theta_0, \phi_0) = 0$ has exactly one solution such that $0 < \theta_0(\phi_0) < \pi$ (see appendix 2) which implies

$$f((b-d)^2, \phi_0) = 0.$$

It can be shown that

$$\begin{aligned}&\rho_1 > 1 > \rho_2 && \text{if } |\phi| < \phi_0\\ &\rho_1, \rho_2 > 1 && \text{if } \pi \geqslant |\phi| > \phi_0 \text{ and } d' > b\\ &\rho_1, \rho_2 < 1 && \text{if } \pi \geqslant |\phi| > \phi_0 \text{ and } d' < b.\end{aligned}$$

It follows from lemma 5 of appendix 2 that

$$\psi = \frac{1}{2\pi}\int_{\phi_0}^{\pi}\mathrm{d}\phi\,\ln\max\{b, d'\}+\frac{1}{2\pi}\int_0^{\phi_0}\mathrm{d}\phi\,\ln(b\rho_1). \tag{A.35}$$

Equation (A.35) reduces to (A.33) since

$$\rho_{1,2} = \frac{1}{2b}[(y+4bd')^{1/2} \pm y^{1/2}].$$

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